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Uniform Properties of Normal and Compact Fuzzy Topological Spaces

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In this paper we continue the study, begun in [1], of fuzzy topological spaces, in particular of some questions connected with the fuzzy uniformities. Since several different approaches to fuzzy topology have been introduced, we point out that the context in which we work is that provided by Hutton, which we consider to be the most satisfactory.

The main results we present are the generalizations to fuzzy spaces of these well-known theorems:

(i) a completely regular space is normal if and only if the family of finite open covers is a basis for a compatible uniformity (Theorem 5);

(ii) a completely regular compact space admits exactly one compatible uniformity (Theorem 9).

The techniques of the proofs are based on the results of [4] concerning the fuzzy products and on some concepts introduced in [1].

A fuzzy lattice is a complete, completely distributive lattice with order reversing involution; its elements are called fuzzy sets: 0 and 1 denote the smallest and greatest elements, respectively.

If L, M are fuzzy lattices, a morphism f from L to M is a function $f^{\leftarrow}$ from M to L which preserves $\bigwedge$, $\bigvee$, $'$, and such that $f^{\leftarrow}(0)=0$.

For the definitions of fuzzy topological space and continuous function, see [4]. We consider the uniform structure introduced in [3]. Denote by $\mathscr{Q}$ the set of the functions $U\colon L\to L$ which verify:

(A1) $U(0)=0$,

(A2) $U(\mu)\geqslant\mu$ for every $\mu\in L$,

(A3) $U(\bigvee\mu_i)=\bigvee U(\mu_i)$ for $\mu_i\in L$.

A uniformity $\mathscr{U}$ is a non-empty subset of $\mathscr{Q}$ such that:

(U1) $U\in\mathscr{U}$ and $U\leqslant V\in\mathscr{Q}$ imply $V\in\mathscr{U}$,

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(U2) $U, V \in \mathcal{U}$ implies $U \wedge V \in \mathcal{U}$,

(U3) $U \in \mathcal{U}$ implies that there exists $V \in \mathcal{U}$ such that $V \circ V \leqslant U$,

(U4) $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$.

The meaning of $U \wedge V$ and U^{-1} is that of [3].

A uniform map $f: (L, \mathcal{U}) \to (M, \mathcal{V})$ is a morphism from L to M such that $f^{\leftarrow}(V)$ belongs to $\mathcal{U}$ for every $V \in \mathcal{V}$, where $f^{\leftarrow}(V)$ is defined by

$$f^{\leftarrow}(V)(\mu) = f^{\leftarrow}(V(f(\mu)))$$

(the definition of $f(\mu)$ is given in [4]: $f(\mu) = \inf\{\nu \in M: \mu \leqslant f^{\leftarrow}(\nu)\}$).

In what follows we shall need some results and concepts of [1]: although the context of [1] was less general, the content of Sections 1, 2, 4, and 5 can be adapted to the present situation without any trouble. As an example we provide the following proposition:

1. Proposition. *Let $(L, \mathcal{U})$, $(M, \mathcal{V})$ be uniform spaces, f a morphism from the fuzzy lattice L to the fuzzy lattice M, and $\mathcal{S}$ a sub-basis of $\mathcal{V}$. Then f is a uniform map if and only if $f^{\leftarrow}(S)$ belongs to $\mathcal{U}$ for every $S \in \mathcal{S}$.*

Proof. The "only if" part is trivial.

For the converse, clearly it is enough to show that $f^{\leftarrow}(S_1 \wedge S_2)$ belongs to $\mathcal{U}$ whenever S_1, S_2 belong to $\mathcal{S}$: we shall show that $f^{\leftarrow}(S_1 \wedge S_2) = f^{\leftarrow}(S_1) \wedge f^{\leftarrow}(S_2)$. Trivially the left member of the equality is less than or equal to the right one, since $U \leqslant V$ implies $f^{\leftarrow}(U) \leqslant f^{\leftarrow}(V)$. For the opposite inequality, take $\mu \in L$ and put $\nu = f(\mu)$; then if $\nu_1 \vee \nu_2 = \nu$, put $\mu_i = \mu \wedge f^{\leftarrow}(\nu_i)$ for $i = 1, 2$; immediately we get $\mu_1 \vee \mu_2 = \mu$ and $\nu_i \geqslant f(\mu_i)$, $i = 1, 2$. Therefore

$$\begin{aligned}
&(f^{\leftarrow}(S_1) \wedge f^{\leftarrow}(S_2))(\mu) \\
&\quad = \inf_{\alpha_1 \vee \alpha_2 = \mu} f^{\leftarrow}(S_1)(\alpha_1) \vee f^{\leftarrow}(S_2)(\alpha_2) \\
&\quad \leqslant \inf_{\nu_1 \vee \nu_2 = \nu} f^{\leftarrow}(S_1)(\mu_1) \vee f^{\leftarrow}(S_2)(\mu_2) \\
&\quad = \inf_{\nu_1 \vee \nu_2 = \nu} f^{\leftarrow}(S_1(f(\mu_1))) \vee f^{\leftarrow}(S_2(f(\mu_2))) \\
&\quad \leqslant \inf_{\nu_1 \vee \nu_2 = \nu} f^{\leftarrow}(S_1(\nu_1)) \vee f^{\leftarrow}(S_2(\nu_2)) \\
&\quad = f^{\leftarrow}\left(\inf_{\nu_1 \vee \nu_2 = \nu} S_1(\nu_1) \vee S_2(\nu_2)\right) = f^{\leftarrow}(S_1 \wedge S_2(\nu)) \\
&\quad = f^{\leftarrow}(S_1 \wedge S_2)(\mu). \quad \blacksquare
\end{aligned}$$

For the definitions of separation and regularity axioms and related properties we shall refer to [5]; see also [2].

Clearly the first part of the third section of [1] is meaningless in the present case; however, Theorem 3.5 has a satisfactory reformulation in:

2. THEOREM. *Let (L, τ) be a fuzzy topological space. Define $\delta(\mu, \rho) = 0$ if and only if $\bar{\mu} \leqslant (\bar{\rho})'$ (for any fuzzy set λ, $\bar{\lambda}$ denotes its closure). Then:*

(i) *δ is a proximity if and only if (L, τ) is normal;*

(ii) *if δ is a proximity, then the topology τ_δ induced by δ is coarser than τ;*

(iii) *if δ is a proximity, then $\tau_\delta = \tau$ if and only if (L, τ) is R_0: clearly in this case δ is the finest proximity which induces τ.*

Proof. (i) and (ii) See [1].

(iii) τ_δ is always R_0 by the definition and 2.8(i) in [1]. Conversely, a τ-open fuzzy set μ is the supremum of a family of τ-closed fuzzy sets, which are δ-far from μ': hence μ is τ_δ-open. ∎

In [4] Hutton has established a lattice isomorphism, denoted by θ, between $L \otimes L$ and the lattice of the maps from L into L which satisfy (A1) and (A3), pointwise ordered: if $U: L \to L$ is such a map, we denote by $\tilde{U}$ the element of $L \otimes L$ such that $\theta(\tilde{U}) = U$; the element corresponding to the identity map is denoted by Δ.

We need two more notations: if μ, ρ are fuzzy sets, $\mu \otimes \rho$ will indicate the "box" determined by the pair (μ, ρ), i.e., $\mu \otimes \rho = \{(\sigma, \tau)\colon 0 < \sigma \leqslant \mu, 0 < \tau \leqslant \rho\}$; as in [1], denote by $U_{\mu\rho}$ the map from L into L so defined:

$$U_{\mu\rho}(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0 \\ \rho' & \text{if } 0 < \lambda \leqslant \mu \\ 1 & \text{otherwise.} \end{cases}$$

Using the correspondence θ, it is easy to see that $\tilde{U}_{\mu\rho} = (\mu \otimes \rho)'$: in fact if σ, τ are different from 0, we have $U_{\mu\rho}(\sigma) \leqslant \tau'$ if and only if $\sigma \leqslant \mu$ and $\tau' \geqslant \rho'$.

3. LEMMA. $\bar{\mu} \otimes \bar{\rho} = \overline{\mu \otimes \rho}$.

Proof. Clearly $\bar{\mu} \otimes \bar{\rho}$ is closed since it is equal to $\pi_1^{\leftarrow}(\bar{\mu}) \wedge \pi_2^{\leftarrow}(\bar{\rho})$, where π_1 and π_2 are the canonical projections. Hence it is enough to show that $(\bar{\mu}, \bar{\rho})$ belongs to every basic closed set greater than $\mu \otimes \rho$. For closed sets σ, τ, we have $\mu \otimes \rho \leqslant \pi_1^{\leftarrow}(\sigma) \vee \pi_2^{\leftarrow}(\tau)$ implies $(\mu, \rho) \in \pi_1^{\leftarrow}(\sigma) \cup \pi_2^{\leftarrow}(\tau)$, say $(\mu, \rho) \in \pi_1^{\leftarrow}(\sigma)$: hence $\bar{\mu} \leqslant \sigma$, which implies $(\bar{\mu}, \bar{\rho}) \in \pi_1^{\leftarrow}(\sigma)$. ∎

For any fuzzy set μ, denote by $\mathrm{Int}(\mu)$ its interior.

4. PROPOSITION. *Let L be a fuzzy topological space and $U: L \to L$ a map belonging to $\mathcal{Q}$. We have*:

(i) *if $\tilde{U}$ is a neighborhood of Δ, then $U(\mu)$ is a neighborhood of $\bar{\mu}$ for every $\mu \in L$;*

(ii) *if the range of U is finite and $U(\mu)$ is a neighbourhood of $\bar{\mu}$ for every $\mu \in L$, then $\tilde{U}$ is a neighbourhood of Δ;*

(iii) *if there exists a symmetric element $V \in \mathcal{Q}$ such that $V \circ V \leqslant U$ and $V(\mu)$ is a neighbourhood of $\bar{\mu}$ for every $\mu \in L$ (in particular if U belongs to a compatible uniformity), then $\tilde{U}$ is a neighbourhood of Δ.*

Proof. (i) For every $\mu \neq 0$, the pair $(\mu, U(\mu)')$ belongs to $\tilde{U}'$; hence $(\bar{\mu}, \overline{U(\mu)'})$ belongs to Δ', that is, $\bar{\mu} \leqslant \overline{U(\mu)'}' = \mathrm{Int}(U(\mu))$.

(ii) Let $\{\rho_1, ..., \rho_n\}$ be the range of U and put $\mu_i = \sup\{\lambda \in L: U(\lambda) \leqslant \rho_i\}$. Since U preserves suprema, $U(\mu_i) = \rho_i$, hence ρ_i is a neighbourhood of $\bar{\mu}_i$. This implies that $\bar{\mu}_i \otimes \overline{\rho_i'} \leqslant \Delta'$: the conclusion follows since $U = \bigwedge U_{\mu_i \rho_i'}$ and $\tilde{}$ is a lattice isomorphism.

(iii) μ, ρ, λ always denote elements of $L \setminus \{0\}$. $\tilde{U}' = \{(\mu, \rho): U(\mu) \leqslant \rho'\}$: hence $\tilde{U}'$ is the smallest element of $L \otimes L$ which contains $\{(\lambda, U(\lambda)'): \lambda \in L \setminus \{0\}\}$, and an easy calculation gives that $\tilde{U} = \{(\mu, \rho): \forall \lambda$ either $\mu \leqslant \lambda'$ or $\rho \leqslant U(\lambda)\}$. In the same way we get that $\Delta = \{(\mu, \rho): \forall \lambda$ either $\mu \leqslant \lambda'$ or $\rho \leqslant \lambda\}$. Clearly $\Delta \leqslant \tilde{U}$; moreover if $(\mu, \rho) \in \Delta$, take V as in the hypothesis and observe that $V(\mu) \otimes V(\rho) \leqslant \tilde{U}$: in fact, for every λ we have either $\mu \leqslant V(\lambda)'$ or $\rho \leqslant V(\lambda)$, which implies either $V(\mu) \leqslant V(V(\lambda)') \leqslant \lambda'$ (see [3, Proposition 10]) or $V(\rho) \leqslant V(V(\lambda)) \leqslant U(\lambda)$, hence $(V(\mu), V(\rho)) \in \tilde{U}$. We conclude by observing that $\bigcup_{(\mu, \rho) \in \Delta} \mathrm{Int}\, V(\mu) \otimes \mathrm{Int}\, V(\rho)$ is an open fuzzy set in $L \otimes L$ which contains Δ and is contained in $\tilde{U}$. ∎

5. THEOREM. *Let L be a fuzzy topological space. L is normal and R_0 if and only if the family $\mathcal{B} = \{U: L \to L$ such that: $U \in \mathcal{Q}$, the range of U is finite and $U(\mu)$ is a neighbourhood of $\bar{\mu}$ $\forall \mu \in L\}$ is a basis for a compatible uniformity.*

Proof. By Theorem 2, if L is normal and R_0, there exists a canonical compatible proximity δ. If $\mathcal{S}$ is the sub-basis of the uniformity $\mathcal{U}_\delta$ described in Section 4 of [1], clearly $\mathcal{S} \subseteq \mathcal{B}$; on the other hand, as in the previous proposition, every $U \in \mathcal{U}$ is the infimum of certain $U_{\mu_i \rho_i'}$, where ρ_i is a neighbourhood of $\bar{\mu}_i$, that is, $\bar{\mu}_i$ and $\overline{\rho_i'}$ are δ-far, namely every $U_{\mu_i \rho_i'}$ belongs to $\mathcal{S}$: hence $\mathcal{B}$ generates $\mathcal{U}_\delta$.

Conversely, take closed fuzzy sets μ, ρ such that $\mu \leqslant \rho'$. Since $U_{\mu\rho}$ belongs to $\mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V \circ V \leqslant U_{\mu\rho}$: hence $\mu \leqslant \mathrm{Int}(V(\mu)) \leqslant \overline{V(\mu)} \leqslant V(V(\mu)) \leqslant U_{\mu\rho}(\mu) = \rho'$, that is, L is normal; it is R_0 by [5]. ∎

Our aim is now to show that a completely regular compact space admits exactly one compatible uniformity, which result extends a classical theorem of general topology. We refer to the definition of fuzzy compact space given by Hutton in [4], namely: a fuzzy topological space is said to be compact if every open cover of a fuzzy closed set has a finite subcover.

6. PROPOSITION. *Let L be a compact uniformizable space, $\mathscr{U}$ a compatible uniformity, and ρ a neighbourhood of a fuzzy closed set μ. Then ρ is a uniform neighbourhood of μ, that is, there exists $U \in \mathscr{U}$ such that $U(\mu) \leqslant \rho$.*

Proof. We have $\operatorname{Int}(\rho) = \sup\{\lambda : \exists U_\lambda \in \mathscr{U}$ such that $U_\lambda(\lambda) \leqslant \rho\}$; take $V_\lambda \in \mathscr{U}$ such that $V_\lambda \circ V_\lambda \leqslant U_\lambda$ and observe that $\{\operatorname{Int} V_\lambda(\lambda)\}$ is an open cover of μ. By compactness there exist $\lambda_1, ..., \lambda_n$ such that $\mu \leqslant \operatorname{Int}(V_{\lambda_1}(\lambda_1)) \vee \cdots \vee \operatorname{Int}(V_{\lambda_n}(\lambda_n))$; the element $V_{\lambda_1} \wedge \cdots \wedge V_{\lambda_n}$ belongs to $\mathscr{U}$ and, owing to a trivial extension of [3, Lemma 3], we can write $(V_{\lambda_1} \wedge \cdots \wedge V_{\lambda_n})(\mu) \leqslant (V_{\lambda_1} \wedge \cdots \wedge V_{\lambda_n})(V_{\lambda_1}(\lambda_1) \vee \cdots \vee V_{\lambda_n}(\lambda_n)) \leqslant U_{\lambda_1}(\lambda_1) \vee \cdots \vee U_{\lambda_n}(\lambda_n) \leqslant \rho$. ∎

7. COROLLARY. *Let L and $\mathscr{U}$ be as before and μ and ρ be closed fuzzy sets such that $\mu \leqslant \rho'$; then $U_{\mu\rho}$ belongs to $\mathscr{U}$.*

Clearly this corollary and the fact that the totally bounded reflection $p\mathscr{U}$ [1] is generated by the elements $U_{\mu\rho}$, where μ, ρ are closed fuzzy sets such that $U(\mu) \leqslant \rho'$ for some $U \in \mathscr{U}$, imply the following:

8. COROLLARY. *Every compact completely regular fuzzy topological space admits a unique totally bounded compatible uniformity.*

Remark. In the above corollary, the hypothesis "completely regular" may be replaced by "R_1": in fact by [5, Propositions 23, 8, 21, 16, and 12], a compact space is R_1 if and only if it is completely regular. In particular such a space is normal and R_0, hence by Theorem 2 and the results of [1] we may describe the unique totally bounded compatible uniformity: it is $\mathscr{U}_\delta$, where δ is the canonical proximity associated to a normal space.

9. THEOREM. *Let L be a compact completely regular space; then there exists a unique compatible fuzzy uniformity.*

Proof. Let Δ_0 be the smallest element containing Δ in the complete complemented sub-lattice of $L \otimes L$ generated by the open subsets. By [5, Proposition 7], Δ_0 is the closure of Δ; hence we have

$$\begin{aligned} \Delta_0' &= \bigvee \{\mu \otimes \rho : \mu, \rho \text{ open sets}, \mu \leqslant \rho'\} \\ &= \bigvee \{\mu \otimes \bar{\mu}' : \mu \text{ open set}\}. \end{aligned}$$

Let $\mathscr{U}$ be a compatible uniformity; if μ is open in the fuzzy topology generated by $\mathscr{U}$, it is immediate to see that μ is the union of the open sets which are $\mathscr{U}$-far from $\bar{\mu}'$, so that $\Delta_0' = \bigvee \{\rho \otimes \mu$: ρ, μ open sets, $\rho\ \mathscr{U}$-far from $\mu\}$. By Proposition 4, for any $U \in \mathscr{U}$, $\tilde{U}$ is a neighbourhood of Δ, hence of Δ_0; by the compactness of $L \otimes L$, we have that there exist ρ_i, μ_i, $i = 1 \cdots n$, such that $\tilde{U} \geqslant \bigwedge_{i=1 \cdots n} (\rho_i \otimes \mu_i)' = \bigwedge_{i=1 \cdots n} \tilde{U}_{\rho_i \mu_i}$, if and only if $U \geqslant \bigwedge_{i=1 \cdots n} U_{\rho_i \mu_i}$, that is, $\mathscr{U}$ is totally bounded. The conclusion now follows from Corollary 8. ∎

10. Corollary. *Any continuous function from the compact completely regular space L into the completely regular space M is uniformly continuous, for every compatible uniformity on M.*

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